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DIFFERENTIABLE CONTROL METRICS AND SCALED BUMP FUNCTIONS

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Abstract

We show that for each control metric (i.e., Carnot-Caratheodory metric), there is an equivalent metric which has the maximal expected degree of smoothness. The equivalent metric satisfies the natural differential inequalities with respect to the vector fields used to define the metric. This generalizes the regularity of the usual Euclidean metric in $\mathbf{R}^n$. There are also corresponding differential inequalities for scaled "bump functions" supported on balls associated to these metrics. The smooth metrics and bump functions are particularly useful in problems of harmonic analysis in situations where the given metrics arise.

1. Introduction

Control metrics are defined in terms of a distinguished family of smooth real vector fields on a manifold M. In a typical situation, one is given a collection $\{X_1, \ldots, X_p\}$ of vector fields on M, and one assumes that these vector fields, together with all their iterated commutators, span the tangent space at each point of M. One defines the *control metric* as follows. The distance between two points is the infimum of the times required to flow from one point to the other along absolutely continuous curves whose tangent, almost everywhere, is a bounded linear combination of the vector fields $X_1, \ldots, X_p$. Such spaces are now often called sub-Riemannian or Carnot-Carathéodory spaces, and they arise in many places in mathematics, including control theory, the theory of hypoelliptic differential operators, and several areas of harmonic and complex analysis. See [1], [11], or [12] for a general discussion and detailed references.

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In many applications, the sizes of distribution kernels of important classes of operators are described in terms of the control metric ρ. However, it is not crucial that one use the control metric itself. Any other metric $\widetilde{\rho}$ such that the ratio $\rho/\widetilde{\rho}$ is bounded and bounded away from zero on compact subsets of $M \times M$ will also suffice. In general, the control metric itself may fail to be differentiable. Thus it is of interest and importance to construct equivalent metrics which have good differentiability properties with respect to the family of control vector fields. It is also important in applications to have smooth functions supported in balls of radius δ which satisfy good differential inequalities with respect to the control vector fields.

The proto-typical example is the case of $\mathbb{R}^n$ with the control vector fields $\{\partial_{x_j}\}$, $1 \leq j \leq n$. The metrics

$$\rho_1(x, y) = \sum_{j=1}^{m} |x_j - y_j| \quad \text{and} \quad \rho_\infty(x, y) = \sup_{1\leq j\leq n} |x_j - y_j|$$

are equivalent to the control metric, but are not smooth. However the equivalent Euclidean metric ρ has particularly good differential properties. ρ is smooth away from the diagonal, and if ∂_x^α, ∂_y^β are derivatives of total order $|\alpha|$ and $|\beta|$ in variables x and y, then

$$\text{(A)} \qquad \left|\partial_x^\alpha \, \partial_y^\beta \rho(x, y)\right| \lesssim \rho(x, y)^{1-|\alpha|-|\beta|}.$$

Moreover, one can easily construct smooth "bump functions" φ_y, supported and bounded by 1 on a ρ ball of radius δ centered at y and identically equal to 1 on the ball of radius $\delta/2$, such that

$$\text{(B)} \qquad \left|\partial_x^\alpha \, \partial_y^\beta \, \varphi_y(x)\right| \lesssim \delta^{-|\alpha|-|\beta|}.$$

One says that because of estimates (A) and (B), the metric ρ and the associated bump functions scale properly relative to the vector fields.

The object of this paper is to show that for any appropriate system of control vector fields, there exist smooth metrics equivalent to the control metric, and good bump functions which scale properly relative to them.

We consider two somewhat different situations. First, for a general class of control vector fields, we show that for *small* distances δ there are smooth metrics equivalent to the control metric, such that the analogue of estimate (A) holds. We also show there are smooth bump functions ϕ_y supported and uniformly bounded on the ball centered at y of radius

δ such that the appropriate analogue of estimate (B) is satisfied. The essential difficulty here is in obtaining uniform estimates in derivatives with respect to the base point y as well as with respect to the variable x. The main results are given in Theorems 3.3.1 and 3.3.2 below.

In certain examples arising in complex analysis, there are control metrics which are defined for *all* distances $\delta > 0$. We show that here too one can find equivalent smooth metrics and good bump functions which scale properly. The main result is given in Theorem 4.4.6 below. These examples are somewhat special in nature. However, the construction of the smooth metric in these cases has the advantage that it is given explicitly in terms of the basic invariants Λ_j and their variants σ_j. These estimates are important in the analysis of the $\bar{\partial}_b$-complex on decoupled domains in $\mathbb{C}^n$, $n \geq 3$, which will be the subject of future papers.

2. General control vectors and metrics

We begin by describing a version of control vector fields which includes the example described in the Introduction as a special case. We then define the associated control metric generated by vector fields.

2.1 Definition and notation

Let $\Omega \subset \mathbb{R}^N$ be a connected open set, and let $Y_1, Y_2, \ldots, Y_q$ be a list, possibly with repetitions, of real $\mathcal{C}^\infty$ vector fields on Ω. Associate to each entry Y_j in this list an integer $d_j \geq 1$, called the *formal degree* of the vector field Y_j.

Definition 2.1.1. The list of vector fields $\{Y_j\}$, $1 \leq j \leq q$ is of **finite homogeneous type** on Ω if

1. For all $1 \leq j, k \leq q$,

$$[Y_j, Y_k] = \sum_{d_l \leq d_j + d_k} c_{j,k}^l(x)\, Y_l$$

   where $c_{j,k}^l \in \mathcal{C}^\infty(\Omega)$, and $[A, B] = AB - BA$ denotes the commutator of the vector fields A and B.

2. At every point $x \in \Omega$, the set of vectors $\{Y_1(x), \ldots, Y_q(x)\}$ spans $\mathbb{R}^N$.

A fundamental example is the one discussed in the Introduction. Suppose that $X_1, \dots, X_p$ are smooth real vector fields on a connected open set $\Omega \subset \mathbb{R}^N$ such that all the iterated commutators of length at most m span the tangent space at each point. If $Y_1, \dots, Y_q$ is a list of all these iterated commutators, one can associate to a vector field Y_j the length d_j of the commutator from which it arises. Condition (1) then follows from the Jacobi identity.

There are also examples in which our more general formulation is required. For example, in dealing with the equation $X_0 + \sum_{j=1}^{p} X_j^2$, one needs to regard the vector field X_0 as having degree 2, and the other vector fields $X_1, \dots, X_p$ as having degree 1. One can then list all the iterated commutators, but give a commutator involving X_0 an appropriate higher formal degree.

Definition 2.1.2. Let $\{Y_1, \dots, Y_q\}$ be a list of vector fields which are of finite homogeneous type on Ω. For each $\delta > 0$ let $C(\delta)$ denote the set of absolutely continuous curves $\varphi : [0,1] \to \Omega$ which satisfy

$$\varphi'(t) = \sum_{j=1}^{q} a_j(t) Y_j\big(\varphi(t)\big) \quad \text{with} \quad |a_j(t)| \le \delta^{d_j}$$

for almost all $t \in [0,1]$. For $x, y \in \Omega$ set

$$\rho(x,y) = \inf\left\{\delta > 0 \,\middle|\, \Big(\exists \varphi \in C(\delta)\Big)\Big(\varphi(0) = x,\ \varphi(1) = y\Big)\right\}.$$

The function ρ is the **control metric** on Ω generated by the vector fields $Y_1, \dots, Y_q$. Let

$$B(x,\delta) = \left\{y \in \Omega \,\middle|\, \rho(x,y) < \delta\right\}$$

be the ball centered at x of radius δ in the control metric. We denote by $\big|B(x,\delta)\big|$ the volume of this ball.

We shall use the following notation. For any ordered p-tuple $J = (j_1, j_2, \dots, j_p)$ of positive integers with $1 \le j_k \le q$, let

$$d(J) = d_{j_1} + d_{j_2} + \cdots + d_{j_p}$$

be the degree of J. Let

$$\partial_Y^J = Y_{j_1}\, Y_{j_2} \cdots Y_{j_p}$$

be the p$^{\text{th}}$-order differential operator given by the product of the indicated vector fields. If $I = \{i_1, \dots, i_N\}$ is an ordered N-tuple of integers with each $i_j \leq q$, let

$$\lambda_I(x) = \det(Y_{i_1}, \dots, Y_{i_N})(x)$$

where we regard each Y_{i_j} as an N-tuple of smooth functions, and λ_I is then the determinant of the corresponding $N \times N$ matrix. Note that Assumption (2) in Definition 2.1.1 implies that for every $x \in \Omega$, there is an N-tuple I so that $\lambda_I(x) \neq 0$.

2.2 Structure of the control metric

It is known that ρ is a metric, and that it makes Ω into a space of homogeneous type in the sense of [3]. To fix the meaning and the notation, we follows the development in [10].

Theorem 2.2.1. *Let $E \subset\subset \Omega$ be a compact set. There are constants $\delta_0 > 0$ and c_1, $c_2 > 1$ so that for all $x, y \in E$ and all $0 < \delta \leq \delta_0$,*

1. $B(x,\delta) \cap B(y,\delta) \neq \emptyset \quad \Rightarrow \quad B(y,\delta) \subset B(x, c_1\,\delta)$.

2. $\big|B(x, c_1\,\delta)\big| \leq c_2\,\big|B(x,\delta)\big|$.

The proof of Theorem 2.2.1 follows from a more detailed local description of the control metric given in [8]. We shall use several results from that paper. The first gives estimates on the derivatives of the determinants λ_J and is essentially Theorem 6 from that paper.

Theorem 2.2.2. *Let $E \subset\subset \Omega$ be a compact subset. There is a constant $\delta_0 > 0$ with the following property. Let $x \in E$, let $0 < \delta \leq \delta_0$, and let $0 < t < 1$. Choose an N-tuple I such that*

$$\big|\lambda_I(x_0)\big|\,\delta^{d(I)} > t\,\max_J \big|\lambda_J(x_0)\big|\,\delta^{d(J)}.$$

If K is a p-tuple and J is an N-tuple, there are constants $\epsilon > 0$ and $C < +\infty$ such that for all $y \in \Omega$ with $\rho(x,y) < \epsilon_0\delta$ we have

$$\big|\partial_Y^K\,\lambda_J(y)\big| \leq C\,t^{-(p+N)}\,\delta^{d(I)-d(J)-d(K)}\big|\lambda_I(x)\big|.$$

Definition 2.2.3. For $x \in \Omega$ and $\delta > 0$, set

$$\Lambda(x, \delta) = \sum_{I} \left|\lambda_I(x)\right| \delta^{d(I)}$$

where the sum is taken over all N-tuples $I = \{i_1, \dots, i_n\}$ with $1 \leq i_j \leq q$.

The volumes of the balls $B(x, \delta)$ can be estimated in terms of the function Λ. The following is Theorem 1 in [8].

Theorem 2.2.4. *Let $E \subset\subset \Omega$ be a compact subset. Then there are constants $0 < C_1 < C_2$ so that for $x \in E$ and $0 < \delta \leq \delta_0$*

$$C_1 \Lambda(x, \delta) \leq \left|B(x, \delta)\right| \leq C_2 \Lambda(x, \delta).$$

Corollary 2.2.5. *Let $E \subset\subset \Omega$ be a compact subset. There is a constant $\alpha_0 < +\infty$ with the following property. Let $0 < \delta \leq \delta_0$ and let $x, y \in E$ with $\rho(x, y) < \delta$. Then*

$$\left|B(x, \delta)\right| \leq \alpha_0 \left|B(y, \delta)\right|.$$

Proof. By Theorem 2.2.4 it suffices to prove an analogous result for $\Lambda(x, \delta)$ and $\Lambda(y, \delta)$. But this follows by expanding the functions $\lambda_I(y)$ in Taylor series about the point x in exponential coordinates, and using the estimates of Theorem 2.2.2. q.e.d.

The next result describes properties of the exponential mapping given by appropriate N-tuples of vector fields. It is contained in Theorem 7 in [8]. We shall use the following notation for cubes in $\mathbb{R}^N$:

$$Q = \left\{(t_1, \dots, t_N) \in \mathbb{R}^N \,\middle|\, |t_j| < 1 \text{ for all } 1 \leq j \leq N\right\};$$
$$Q^* = \left\{(t_1, \dots, t_N) \in \mathbb{R}^N \,\middle|\, |t_j| < \frac{1}{2} \text{ for all } 1 \leq j \leq N\right\}.$$

Theorem 2.2.6. *Let $E \subset\subset \Omega$ be a compact subset. There exist constants $0 < \eta_0 < \tau_0 < 1$ with the following properties. Let $x \in E$, let $0 < \delta \leq \delta_0$, let $0 < \eta \leq \eta_0$ and let $I = \{i_1, \dots, i_N\}$ be an N-tuple such that*

$$\left|\lambda_I(x)\right| \delta^{d(I)} \geq \tau_0 \max_J \left|\lambda_J(x)\right| \delta^{d(J)}.$$

Define $\Phi_{[I,\delta,\eta,x]} = \Phi : Q \to \Omega$ *by*

$$\Phi(t_1, \dots, t_N) = \exp\left[\eta \sum_{j=1}^{N} \delta^{d_{i_j}}\, t_j\, Y_{i_j}\right](x).$$

Then Φ *is one-to-one and non-singular on* Q*, and if* $|J\Phi|$ *denotes the Jacobian determinant of the mapping, then on* Q

$$\frac{1}{4}\,\eta^N \left|\lambda_I(x)\right| \delta^{d(I)} \le |J\Phi| \le 4\,\eta^N \left|\lambda_I(x)\right| \delta^{d(I)}.$$

Note that Φ is a smooth mapping from the cube $Q \subset \mathbb{R}^N$ onto a set $\Phi(Q)$ containing x_0. The differential $d\Phi$ then carries the derivatives $\{\partial_{t_1}, \dots, \partial_{t_N}\}$ to vector fields on $\Phi(Q)$. The following result is a consequence of Lemma 2.13 in [8].

Theorem 2.2.7. *Let* $E \subset\subset \Omega$ *be a compact subset. Let* $\kappa > 0$*. Then there exists* $\eta = \eta(\kappa)$ *with the following property. Let* $x \in E$ *and let* $0 < \delta < \delta_0$*. Choose an* N*-tuple* $I = \{i_1, \dots, i_N\}$ *satisfying*

$$\left|\lambda_I(x)\right| \delta^{d(I)} \ge \tau_0 \max_J \left|\lambda_J(x)\right| \delta^{d(J)}.$$

Then if $\Phi = \Phi_{[I,\delta,\eta(\kappa),x]}$*,*

$$d\Phi\left(\partial_{t_j}\right) = \eta\, \delta^{d_{i_j}}\, Y_{i_j} + \eta \sum_{\ell=1}^{N} b_{j,\ell}(x)\, Y_{i_\ell},$$

where $\left|b_{j,\ell}\right| \le \kappa\, \delta^{d(Y_{i_\ell})}$ *on* $\Phi(Q)$*.*

The final result gives an alternative description of the balls defined by the control metric. It is contained in Theorem 3 of [8].

Theorem 2.2.8. *Let* $E \subset\subset \Omega$ *be a compact subset. There are constants* $0 < \epsilon_0 < 1 < \sigma_0$ *with the following property. Let* $x \in \Omega$ *and let* $0 < \delta \le \delta_0$*. Choose an* N*-tuple* $I = \{i_1, \dots, i_N\}$ *satisfying*

$$\left|\lambda_I(x)\right| \delta^{d(I)} \ge \tau_0 \max_J \left|\lambda_J(x)\right| \delta^{d(j)}.$$

Then if $\Phi = \Phi_{[I,\delta,\eta,x]}$ *is defined as in Theorem 2.2.6,*

$$\left\{y \in \Omega \,\middle|\, \rho(x,y) < \epsilon_0\, \delta\right\} \subset \Phi(Q^*) \subset \Phi(Q) \subset \left\{y \in \Omega \,\middle|\, \rho(x,y) < \sigma_0\, \delta\right\}.$$

3. The construction of smooth metrics and scaled bump functions

3.1 A preliminary bump functions

We begin by constructing a function $\varphi_x(y)$ supported in a ball centered at x of radius comparable to δ which satisfies the correctly scaled differential inequalities in the variable y. Note that we do not yet obtain estimates on derivatives with respect to the center x.

Lemma 3.1.1. *Let $E \subset\subset \Omega$ be a compact set. Then for every p-tuple $K = \{k_1, \ldots, k_p\}$ there is a constant C_K with the following properties. Let $x \in E$ and let $0 < \delta \leq \delta_0$. Then there exists a function $\varphi = \varphi_{x,\delta} \in \mathcal{C}_0^\infty(\Omega)$ such that:*

1. *For all $y \in \Omega$, $\quad 0 \leq \varphi(y) \leq 1$.*
2. *If ϵ_0 and σ_0 are the constants from Theorem 2.2.8, then*
$$\begin{aligned} \varphi(y) &\equiv 0 \qquad \text{if } \rho(x,y) > \sigma_0\,\delta; \\ \varphi(y) &\equiv 1 \qquad \text{if } \rho(x,y) < \epsilon_0\,\delta. \end{aligned}$$
3. *For every p-tuple K,*
$$\sup_{y\in\Omega}\left|\partial_Y^K \varphi(y)\right| \leq C_K\,\delta^{-d(K)}.$$

Proof. Choose a function $\psi \in \mathcal{C}_0^\infty(Q)$ with $\psi(t) \equiv 1$ if $t \in Q^*$. Given $x \in \Omega$ and $0 < \delta \leq \delta_0$, choose an N-tuple I so that the conclusions of Theorems 2.2.2 through 2.2.8 are true. With κ to be chosen later, let $\Phi = \Phi_{[I,\delta,\eta(\kappa),x]}$ be the diffeomorphism of Theorem 2.2.8. Put $\varphi = \psi \circ \Phi^{-1}$. According to Theorem 2.2.8, $\varphi(y) \equiv 1$ if $\rho(x,y) < \epsilon_0\,\delta$, while $\varphi(y) \equiv 0$ if $\rho(x,y) \geq \sigma_0\,\delta$. On the other hand, it follows from Theorem 2.2.7 that for κ sufficiently small depending only on the compact set E, there are smooth functions $\alpha_{j,m}$ on Q, with $|\alpha_{j,m}(t)| \leq 1$, so that if
$$T_j = \sum_{m=1}^N \alpha_{j,m}(t)\,\partial_{t_m}$$
then
$$d\Phi(T_j) = \delta^{d_{i_j}}\,Y_{i_j}.$$

It follows that

$$\sup_{y\in\Omega}\left|\partial_Y^K\varphi(y)\right| \leq \delta^{-d(K)}\sup_{|\beta|\leq p}\left|\partial_t^\beta\psi(t)\right|.$$

This completes the proof. q.e.d.

3.2 Preliminary distance functions

We next construct smooth, properly scaled functions $\omega_\delta(x,y)=\omega(x,y)$ which vanish when $\rho(x,y)\geq\delta$, and are bounded away from zero when $0\leq\rho(x,y)<<\delta$. To do this, we need the following elementary result about overlapping balls.

Proposition 3.2.1. *Let $K>1$ and let $E\subset\subset\Omega$ be compact. Then there is a positive integer M so that if $0<\delta\leq\delta_0$ and $\{B(x_j,\delta)\}$ is a disjoint collection of balls, then no point of E lies in more than M of the dilated balls $\{B(x_j,K\,\delta)\}$.*

Proof. Let $y\in E$, and suppose that $y\in B(x_{j_\ell},K\,\delta)$ for $\ell=1,\dots,M$. Since $y\in B(y,K\,\delta)\cap B(x_{j_\ell},K\,\delta)$, it follows from Theorem 2.2.1 that

$$B(x_{j_\ell},\delta)\subset B(x_{j_\ell},K\,\delta)\subset B(y,c_1K\,\delta).$$

Since the balls $\{B(x_j,\delta)\}$ are disjoint, we have

$$\left|B(y,c_1K\,\delta)\right|\geq\sum_{\ell=1}^{M}\left|B(x_{j_\ell},\delta)\right|.$$

But by Theorem 2.2.1, there is a constant $\beta_0<+\infty$ depending on the structure constants c_1 and c_2 and on the constant K so that

$$\left|B(x_{j_\ell},\delta)\right|\geq\beta_0^{-1}\left|B(x_{j_\ell},c_1K\,\delta)\right|$$

and since $\rho(x_{j_\ell},y)\leq K\,\delta$, it follows from Corollary 2.2.5 that

$$\left|B(x_{j_\ell},c_1K\,\delta)\right|\geq\alpha_0^{-1}\left|B(y,c_1K\,\delta)\right|.$$

Thus it follows that

$$M\leq\alpha_0\,\beta_0$$

which is a constant that depends only on the structural constants c_1 and c_2, on K, and on the compact set E. (See [10, p. 32], for a similar argument.) This completes the proof. q.e.d.

Lemma 3.2.2. *Let $E \subset\subset \Omega$ be a compact set. There are constants $0 < \gamma_0$, $M < +\infty$, and $C_{K,L}$ for every pair of tuples $K = \{k_1, \dots, k_p\}$, $L = \{l_1, \dots, l_r\}$ with the following properties. Let $0 < \delta \leq \delta_0$. Then there exists a function $\omega = \omega_\delta \in \mathcal{C}^\infty(\Omega \times \Omega)$ such that for any $x, y \in E$:*

1. $$0 \leq \omega(x, y) \leq M.$$

2. $$\begin{aligned} \omega(x, y) \geq 1 \quad & \textit{if } \rho(x, y) \leq \delta \\ \omega(x, y) = 0 \quad & \textit{if } \rho(x, y) \geq \gamma_0\, \delta. \end{aligned}$$

3. *Given a p-tuple K and a r-tuple L*
$$\sup_{(x,y) \in E \times E} \left| \partial_Y^K\, \partial_Y^L\, \omega(x, y) \right| \leq C_{K,L}\, \delta^{-d(K) - d(L)}$$
where ∂_Y^K acts on the variable x and ∂_Y^L acts on the variable y.

Proof. Cover the set E by balls $B(x, \delta)$, $x \in E$, and choose a maximal subcollection of these balls $\{B(x_j, \delta)\}$ so that $B(x_j, \delta) \cap B(x_k, \delta) = \emptyset$ for $j \neq k$. Let $y \in E$ be arbitrary. If $y \notin \bigcup_j B(x_j, \delta)$, then by the maximality of the subcollection, the ball $B(y, \delta)$ must intersect one of the balls $B(x_k, \delta)$. But then according to Theorem 2.2.1, $B(y, \delta) \subset B(x_k, c_1\, \delta)$, and hence it follows that the collection of balls $\{B(x_j, c_1 \delta)\}$ is a cover of E. Suppose $x, y \in E$ with $\rho(x, y) \leq \delta$. There is an index k so that $x \in B(x_k, c_1\, \delta)$ and hence $y \in B(x_k, 2c_1\, \delta)$. Thus given any two points $x, y \in E$ with $\rho(x, y) \leq \delta$, there is an index k with $x, y \in B(x_k, 2c_1\, \delta)$.

According to Lemma 3.1.1 there is a function $\varphi_j \in \mathcal{C}_0^\infty(\Omega)$ so that $0 \leq \varphi_j(y) \leq 1$ for all $y \in \Omega$ and
$$\varphi_j(y) = \begin{cases} 1 & \text{if } \rho(x_j, y) \leq 2c_1\, \delta \\ \\ 0 & \text{if } \rho(x_j, y) \geq 2c_1\, \sigma_0\, \epsilon_0^{-1} \delta \end{cases}$$
$$\left| \partial_Y^K \varphi_j(y) \right| \leq C_K\, \delta^{-d(K)} \quad \text{for all } y \in \Omega.$$

Consider the function

$$\omega(x,y) = \sum_j \varphi_j(x)\varphi_j(y).$$

Since all the functions φ_j are non-negative, it follows that if $\omega(x,y) \neq 0$, then at least one of the products $\varphi_j(x)\varphi_j(y) \neq 0$. This implies that $x, y \in B(x_j, 2c_1\,\sigma_0\,\epsilon_0^{-1}\delta)$ and hence that $\rho(x,y) < 4c_1\,\sigma_0\,\epsilon_0^{-1}\delta$. Put $\gamma_0 = 4c_1\,\sigma_0\,\epsilon_0^{-1}$. Then

$$\rho(x,y) > \gamma_0\,\delta \Rightarrow \omega(x,y) \equiv 0.$$

On the other hand, if $\rho(x,y) < \delta$, there is an index j so that $\varphi_j(x) = \varphi_j(y) = 1$. Hence

$$\rho(x,y) < \delta \Rightarrow \omega(x,y) \geq 1.$$

This proves Condition (2).

For any fixed x, the only terms in the sum for $\omega(x,y)$ which are not zero come from balls $B(x_j,\delta)$ which are disjoint but such that the dilates $B(x_j, 2c_1\,\sigma_0\,\epsilon_0^{-1}\delta)$ all contain x. By Proposition 3.2.1, there are only M such balls, where M is a constant depending only on the compact set E. Since each φ_j is bounded by 1, it follows that $\omega(x,y) \leq M$. This proves Condition (1). Condition (3), the estimates on the derivatives of ω, now follow from the estimates in Lemma 3.1.1. This completes the proof of Lemma 3.2.2. q.e.d.

Remark 3.2.3. Variants of this result appear in [9] and in [4] or [5].

3.3 The main results for small distances

Theorem 3.3.1. *Let $E \subset\subset \Omega$ be compact. Then there is a function $\widetilde{\rho}$ defined on $\Omega \times \Omega$ which is smooth away from the diagonal such that:*

1. *For all $x \neq y \in E$, the ratio $\rho(x,y)/\widetilde{\rho}(x,y)$ is bounded and bounded away from zero.*

2. *Given a p-tuple K and a r-tuple L*

$$\left|\partial_Y^K\,\partial_Y^L\widetilde{\rho}(x,y)\right| \lesssim \widetilde{\rho}(x,y)^{1-d(K)-d(L)}.$$

Proof. We only need to define $\widetilde{\rho}(x,y)$ when $\rho(x,y) \leq \delta_0$. According to Lemma 3.2.2, we can find functions $\{\omega_j\}$ on $\Omega \times \Omega$ such that $\omega_j(x,y) \leq M$ for all x, y, and such that

$$\begin{aligned} \omega_j(x,y) &\geq 1 \quad \text{when } 0 \leq \rho(x,y) \leq 2^{-j}\,\delta_0 \\ \omega_j(x,y) &\equiv 0 \quad \text{when } \rho(x,y) \geq \gamma_0\, 2^{-j}\,\delta_0. \end{aligned}$$

Put $\widetilde{\rho}(x,x) = 0$, and for $x \neq y$ put

$$\widetilde{\rho}(x,y) = \left[\sum_{j=0}^{\infty} 2^j\, \omega_j(x,y)\right]^{-1}.$$

Because of the support conditions on ω_j, it is easy to check that

$$\sum_{j=0}^{\infty} 2^j\, \omega_j(x,y) \approx \rho(x,y)^{-1},$$

which proves (1). Since $\left|\partial_Y^K\, \omega_j(x,y)\right| \leq C_K \left(2^{-j}\,\delta_0\right)^{-d(K)}$, it is also easy to check the differential inequality (2). This completes the proof. q.e.d.

Theorem 3.3.2. *Let $E \subset\subset \Omega$ be compact, and let $0 < \delta \leq \delta_0$. Then there is a smooth function ψ on $\Omega \times \Omega$ so that for $x, y \in E$*

1. *$\psi(x,y) \equiv 1$ if $\widetilde{\rho}(x,y) \leq \delta$ and $\psi(x,y) \equiv 0$ if $\widetilde{\rho}(x,y) \geq 2\,\delta$.*
2. *Given a p-tuple K and a r-tuple L*

$$\left|\partial_Y^K\, \partial_Y^L \psi(x,y)\right| \lesssim \delta^{-d(K)-d(L)}.$$

Proof. Let $\widetilde{\rho}$ be the function constructed in Theorem 3.3.1, and let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with $\chi(t) \equiv 1$ if $t \leq a$ and $\chi(t) \equiv 0$ if $t \geq b$ with $a < b$. Put

$$\psi(x,y) = \chi\left(\frac{\widetilde{\rho}(x,y)}{\delta}\right).$$

Then the equivalence of ρ and $\widetilde{\rho}$ in part (1) and the estimates on the derivatives of $\widetilde{\rho}$ in part (2) of Theorem 3.3.1 together with the support properties of χ and χ' show that ψ satisfies conditions (1) and (2) for a suitable choice of a and b. This completes the proof. q.e.d.

4. Metrics on model pseudoconvex boundaries

In this section we construct smooth global control metrics on certain manifolds which arise as the boundary of model pseudoconvex domains of finite type in $\mathbb{C}^2$.

4.1 Definitions

Let P be a non-harmonic polynomial of degree m on $\mathbb{C}$. Let

$$M = \{(z,w) \in \mathbb{C}^2 \,\big|\, \Im\mathrm{m}[w] = P(z)\}.$$

If we set $Z = \dfrac{\partial}{\partial z} + 2i\,\dfrac{\partial P}{\partial z}\,\dfrac{\partial}{\partial w}$ and $\overline{Z} = \dfrac{\partial}{\partial \bar{z}} - 2i\,\dfrac{\partial P}{\partial \bar{z}}\,\dfrac{\partial}{\partial \overline{w}}$, we can consider the control metric for the vector fields $\{\Re\mathrm{e}(\overline{Z}), \Im\mathrm{m}(\overline{Z}), T\}$ where $T = \dfrac{\partial}{\partial w} + \dfrac{\partial}{\partial \overline{w}}$ is a transverse vector field. If P is sub-harmonic so that M is the boundary of a pseudoconvex domain of finite type, this metric controls the size of the Szegö kernel and the relative fundamental solution for $\Box_b$ on M. (See [6], [7], and [2] for background material.)

We can identify M with $\mathbb{C} \times \mathbb{R}$ so that the point $(z,t) \in \mathbb{C} \times \mathbb{R}$ corresponds to $(z, t + iP(z)) \in M$. Under this identification, the vector fields Z and $\overline{Z}$ become

$$Z = \frac{\partial}{\partial z} + \imath \frac{\partial P}{\partial z}\,\frac{\partial}{\partial t}$$

and

$$\overline{Z} = \frac{\partial}{\partial \bar{z}} - \imath \frac{\partial P}{\partial \bar{z}}\,\frac{\partial}{\partial t}.$$

For $z \in \mathbb{C}$ and $2 \leq k \leq m$ define

$$\Lambda_k(z) = \sum_{\alpha+\beta=k-2} \left| \frac{\partial^{\alpha+\beta+2} P}{\partial z^{\alpha+1}\,\partial \bar{z}^{\beta+1}}(z) \right|$$

and then $\Lambda(z,\delta) = \displaystyle\sum_{k=2}^{m} \Lambda_k(z)\,\delta^k$.

The control metric ball centered at the point $(w,s) \in \mathbb{C}\times\mathbb{R}$ of radius $\delta > 0$ is then given up to constants by

$$B\big((w,s),\delta\big) \approx \Bigg\{ (z,t) \in \mathbb{C} \times \mathbb{R} \,\Bigg|\, |z-w| < \delta, \quad \text{and} \quad \left| t - s + 2\,\Im\mathrm{m}\left[\sum_{j=1}^{m} \frac{1}{j!}\,\frac{\partial^j P}{\partial z^j}(w)(z-w)^j \right] \right| < \Lambda(w,\delta) \Bigg\}.$$

If as usual, we define the inverse to the function $\delta \to \Lambda(z,\delta)$ by $s \to \mu(z,s)$, the non-isotropic control distance from the point (z,t) to the base point (w,s) is given up to constants by

$$d\big((z,t),(w,s)\big)$$
$$\approx |z-w| + \mu\left(w, \left|t - s + 2\,\Im\mathrm{m}\left[\sum_{j=1}^{m} \frac{1}{j!}\frac{\partial^j P}{\partial z^j}(w)(z-w)^j\right]\right|\right).$$

The functions d, Λ and μ in this definition are not smooth, and our objective is to construct a smooth variant D which is *globally* equivalent to d and which satisfies good differential inequalities with respect to the vector fields Z and $\overline{Z}$.

4.2 Construction and properties of an auxiliary function Δ

Definition 4.2.1.

1. For $z \in \mathbb{C}$ and $2 \le k$ set

$$\sigma_k(z) = \sum_{\alpha+\beta=k-2} \Big|\frac{\partial^{\alpha+\beta+2}P}{\partial z^{\alpha+1}\,\partial \bar{z}^{\beta+1}}(z)\Big|^2.$$

   Note that $\sigma_k(z)$ is a polynomial of degree $2m-2k$.

2. For $\delta \ge 0$ set

$$\sigma(z,\delta) = \sum_{k=2}^{m} \sigma_k(z)\,\delta^{2k}.$$

3. For fixed z, the function $\sigma(z,\delta)$ is monotone increasing and smooth in δ for $\delta \ge 0$. Hence an inverse function exists. The inverse function to $\sigma(z,\delta)$ is denoted by $\tau(z,s)$ so that

$$\sigma\big(z,\tau(z,s)\big) = s \qquad \text{and} \qquad \tau\big(z,\sigma(z,\delta)\big) = \delta.$$

4. Given two points (z,t) and (w,s) in $\mathbb{C}\times\mathbb{R}$, define

$$\widetilde{\Delta}\big((z,t),(w,s)\big)$$
$$= \sigma\big(w,|z-w|\big) + \left|t - s + 2\,\mathrm{Im}\left[\sum_{j=1}^{m} \frac{1}{j!}\frac{\partial^j P}{\partial z^j}(w)\,(z-w)^j\right]\right|^2.$$

5. Finally, given two points (z,t) and (w,s) in $\mathbb{C}\times\mathbb{R}$, define

$$\Delta\big((z,t),(w,s)\big) = \widetilde{\Delta}\big((z,t),(w,s)\big) + \widetilde{\Delta}\big((w,s),(z,t)\big).$$

Proposition 4.2.2. *We have $\sigma(z,\delta) \approx \Lambda(z,\delta)^2$ and $\tau(z,s) \approx \mu(z,\sqrt{s})$. Each function σ_k is a polynomial in z and $\bar{z}$ on $\mathbb{C}$. The function Δ is a polynomial in $\big((z,\bar{z},t),(w,\overline{w},s)\big)$ on $(\mathbb{C}\times\mathbb{R})\times(\mathbb{C}\times\mathbb{R})$.*

Proof. The statements are clear from the formulas since P is a (real analytic) polynomial on $\mathbb{C}$ of degree m. q.e.d.

Proposition 4.2.3. *Given any positive integers r and s, there is a constant $C_{r,s} > 0$ so that*

$$\left|\frac{\partial^{r+s}\sigma_k}{\partial z^r \partial \bar{z}^s}(z)\right| \leq C_{r,s} \sum_{\substack{p+q=2k+r+s \\ p\leq m,\, q\leq m}} \sqrt{\sigma_p(z)\,\sigma_q(z)}.$$

Proof. By Leibniz' rule we have

$$\begin{aligned}(4.2.1)\quad & \frac{\partial^{r+s}\sigma_k}{\partial z^r \partial \bar{z}^s}(z) \\ &= \sum \binom{r}{r_1}\binom{s}{s_1} \frac{\partial^{\alpha+r_1+\beta+s_1}P}{\partial z^{\alpha+r_1}\,\partial \bar{z}^{\beta+s_1}}(z)\cdot \overline{\frac{\partial^{\alpha+r_2+\beta+s_2}P}{\partial z^{\alpha+r_2}\,\partial \bar{z}^{\beta+s_2}}}(z)\end{aligned}$$

where the sum is taken over integers $\alpha, \beta, r_1, r_2, s_1, s_2$ such that $1 \leq \alpha$, $1 \leq \beta$, $\alpha + \beta = k$, and where $r_1 + r_2 = r$ and $s_1 + s_2 = s$. Taking absolute values gives the desired result. q.e.d.

Proposition 4.2.4. *There is a constant $C_k > 0$ so that for $z, w \in \mathbb{C}$ we have*

$$\big|\sigma_k(z) - \sigma_k(w)\big| \leq C_k \sum_{j=1}^{2m} \sum_{\substack{p+q=2k+j \\ p\leq m,\, q\leq m}} \sqrt{\sigma_p(z)\,\sigma_q(z)}\,|z-w|^j.$$

Proof. Expand $\sigma_k(w)$ in a Taylor series about z The series is finite since σ_k is a polynomial of degree $2m - 2k$. The constant term is $\sigma_k(z)$. The higher order terms are expressions of the form

$$\frac{1}{r!\,s!}\,\frac{\partial^{r+s}\sigma_k}{\partial z^r\,\partial \bar{z}^s}(z)\,(w-z)^r\,(\overline{w}-\bar{z})^s$$

with $r + s \geq 1$. Proposition 4.2.3 then gives the desired estimate.

q.e.d.

Proposition 4.2.5. *There is a constant C so that for all (z,t), $(w,s) \in \mathbb{C}\times\mathbb{R}$*

$$\widetilde{\Delta}\big((z,t),(w,s)\big) \leq C\,\widetilde{\Delta}\big((w,s),(z,t)\big)$$

and in particular

$$\Delta\big((z,t),(w,s)\big) \approx \widetilde{\Delta}\big((z,t),(w,s)\big) \approx \widetilde{\Delta}\big((w,s),(z,t)\big).$$

Proof. It suffices to show that

$$\sum_{k=2}^{m} \sigma_k(w)\,|z-w|^{2k} \leq C \sum_{k=2}^{m} \sigma_k(z)\,|z-w|^{2k}$$

and that

$$\begin{aligned}\Bigg|(t-s)+&2\mathrm{Im}\left[\sum_{j=1}^{m}\frac{1}{j!}\,\frac{\partial^j P}{\partial z^j}(w)\,(z-w)^j\right]\Bigg|^2 \\ &\leq \Bigg|(s-t)+2\mathrm{Im}\left[\sum_{j=1}^{m}\frac{1}{j!}\,\frac{\partial^j P}{\partial z^j}(z)\,(w-z)^j\right]\Bigg|^2 \\ &\qquad + \sum_{k=2}^{m}\big[\sigma_k(z)+\sigma_k(w)\big]\,|z-w|^{2k}.\end{aligned}$$

The first estimate follows easily from Proposition 4.2.4. To establish the second estimate, observe that

$$\begin{aligned}&\sum_{j=1}^{m}\frac{1}{j!}\,\frac{\partial^j P}{\partial z^j}(w)\,(z-w)^j \\ &\quad= \sum_{j=1}^{m}\frac{1}{j!}\left[\sum_{k=j}^{m}\frac{1}{(k-j)!}\,\frac{\partial^k P}{\partial z^k}(z)\,(w-z)^{k-j}+E_j(z,w)\right](z-w)^j \\ &\quad= \sum_{k=1}^{m}\frac{\partial^k P}{\partial z^k}(z)\left[\sum_{j=1}^{k}\frac{1}{j!(k-j)!}\,(z-w)^j\,(w-z)^{k-j}\right] \\ &\qquad+\sum_{j=1}^{m}\frac{1}{j!}\,E_j(z,w)\,(z-w)^j\end{aligned}$$

$$= \sum_{k=1}^{m} \frac{1}{k!} \frac{\partial^k P}{\partial z^k}(z) \left[\sum_{j=1}^{k} \binom{k}{j} (w-z)^{k-j} (z-w)^j \right]$$
$$+ \sum_{j=1}^{m} \frac{1}{j!} E_j(z,w)\,(z-w)^j$$
$$= -\sum_{k=1}^{m} \frac{1}{k!} \frac{\partial^k P}{\partial z^k}(z)\,(w-z)^k + \sum_{j=1}^{m} \frac{1}{j!} E_j(z,w)\,(z-w)^j.$$

Here the error term $E_j(z,w)$ can be written as a sum of terms involving *mixed* derivatives of P times appropriate powers of $(z-w)$ and $(\overline{z}-\overline{w})$. These error terms can be dominated by a constant times $\sum_{k=2}^{m}\big[\sigma_k(z)+\sigma_k(w)\big]\,|z-w|^{2k}$. This completes the proof. q.e.d.

Proposition 4.2.6. *We have*

$$\Delta\big((z,t),(w,s)\big) \approx \sigma\big(w, d\big((z,t),(w,s)\big)\big).$$

Proof. This is clear from the definitions and Proposition 4.2.5.
q.e.d.

Proposition 4.2.7. Δ *is a pseudometric on* $\mathbb{C}\times\mathbb{R}$. *This means:*

1. $\Delta\big((z,t),(w,s)\big) \geq 0$, *and that* $\Delta\big((z,t),(w,s)\big) = 0$ *if and only if* $(z,t)=(w,s)$.

2. $\Delta\big((z,t),(w,s)\big) = \Delta\big((w,s),(z,t)\big)$.

3. *There is a constant* $C>0$ *so that if* (z,t), (w,s) *and* (u,r) *are three points in* $\mathbb{C}\times\mathbb{R}$ *then*

$$\Delta\big((z,t),(u,r)\big) \leq C\left[\Delta\big((z,t),(w,s)\big) + \Delta\big((w,s),(u,r)\big)\right].$$

Proof. The first two statements are obvious from the definition, so the main content of this proposition is the generalized triangle inequality (3). Moreover, it suffices to show that (3) holds with Δ replaced by $\widetilde{\Delta}$. We may assume without loss of generality that $|z-w| \leq |w-u|$ It follows that $|z-u| \leq 2|w-u|$. We have

$$\begin{aligned}\sum_{j=2}^{m}\big[\sigma_j(z)+\sigma_j(u)\big]\,|z-u|^{2j} &\lesssim \sum_{j=2}^{m}\big[\sigma_j(z)+\sigma_j(u)\big]\,|w-u|^{2j}\\ &\leq \sum_{j=2}^{m}\big[\sigma_j(z)\big]\,|w-u|^{2j}\\ &\quad + \sum_{j=2}^{m}\big[\sigma_j(w)+\sigma_j(u)\big]\,|w-u|^{2j}.\end{aligned}$$

For the first term on the right we have by Proposition 4.2.4 that

$$\begin{aligned}\sum_{j=2}^{m}\sigma_j(z)\,|w-u|^{2j} &\leq C\sum_{j=2}^{m}\sum_{k=j}^{m}\sigma_k(w)\,|z-w|^{2k-2j}\,|w-u|^{2j}\\ &\lesssim \sum_{k=2}^{m}\sigma_k(w)\,|w-u|^{2k}\\ &\leq \sum_{k=2}^{m}\big[\sigma_k(w)+\sigma_k(u)\big]\,|w-u|^{2k}.\end{aligned}$$

This shows that

$$\sum_{j=1}^{m}\sigma_j(u)\,|z-u|^{2j} \leq C\,\Big[\sum_{j=1}^{m}\sigma_j(w)\,|z-w|^{2j} + \sum_{j=1}^{m}\sigma_j(u)\,|w-u|^{2j}\Big].$$

Next we deal with the other part of $\widetilde{\Delta}$. We have

$$\begin{aligned}&\left|(t-r)+2\mathrm{Im}\sum_{j=1}^{m}\frac{1}{j!}\frac{\partial^j P}{\partial z^j}(u)\,(z-u)^j\right|\\ &\quad = \left|(t-r)+2\mathrm{Im}\sum_{j=1}^{m}\frac{1}{j!}\frac{\partial^j P}{\partial z^j}(u)\,\sum_{k=0}^{j}\frac{j!}{k!(j-k)!}(z-w)^k(w-u)^{j-k}\right|\end{aligned}$$

$$= \Big|(t-r) + 2\mathrm{Im}\sum_{j=1}^{m}\frac{1}{j!}\frac{\partial^j P}{\partial z^j}(u)\,(w-u)^j$$

$$+ 2\mathrm{Im}\sum_{j=1}^{m}\frac{1}{j!}\frac{\partial^j P}{\partial z^j}(u)\,\sum_{k=1}^{j}\frac{j!}{k!(j-k)!}(z-w)^k(w-u)^{j-k}\Big|$$

$$\leq \left|(s-r) + 2\mathrm{Im}\sum_{j=1}^{m}\frac{1}{j!}\frac{\partial^j P}{\partial z^j}(u)\,(w-u)^j\right|$$

$$+ \left|(t-s) + 2\mathrm{Im}\sum_{k=1}^{m}\frac{1}{k!}\frac{\partial^k P}{\partial z^k}(w)(z-w)^k\right|$$

$$+ \left|2\mathrm{Im}\sum_{k=1}^{m}\frac{1}{k!}E_k(z,w)\,(z-w)^k\right|.$$

As before, the error term can be absorbed in the terms that involve mixed derivatives of P. This completes the proof. q.e.d.

4.3 Differential properties of the function Δ

We want to study the effect of applying the "good" vector fields Z and $\overline{Z}$ to the function Δ. Our object is to prove:

Theorem 4.3.1. *Let $Q(Z,\overline{Z})$ be a non-commuting polynomial of degree N in the vector fields Z and $\overline{Z}$, each acting either in the (z,t) variables or in the (w,s) variables. There is a constant C_Q so that*

$$\Big|Q(Z,\overline{Z})\Delta\big((z,t),(w,s)\big)\Big| \leq C_Q\, d\big((z,t),(w,s)\big)^{-N}\,\Delta\big((z,t),(w,s)\big).$$

Note that from the definition of Δ and $\widetilde{\Delta}$, and the symmetry of derivatives in the statement of Theorem 4.3.1, it suffices to establish

$$\Big|Q(Z,\overline{Z})\widetilde{\Delta}\big((z,t),w,s)\big)\Big| \leq C_Q\, d\big((z,t),(w,s)\big)^{-N}\,\widetilde{\Delta}\big((z,t),(w,s)\big)$$

To simplify our formulas, write

$$\begin{aligned}
P_{(j,k)}(z) &= \frac{\partial^{j+k} P}{\partial z^j\,\partial \bar{z}^k}(z);\\
\Psi(z,w,t,s) &= t - s + 2\mathrm{Im}\left[\sum_{j=1}^{m} \frac{1}{j!}\,P_{(j,0)}(w)\,(z-w)^j\right]\\
&= t - s - \imath\left[\sum_{j=1}^{m} \frac{1}{j!}\,P_{(j,0)}(w)\,(z-w)^j\right.\\
&\qquad\qquad \left. - \sum_{j=1}^{m} \frac{1}{j!}\,P_{(0,j)}(w)\,(\bar{z}-\overline{w})^j\right];\\
\Phi(z,w) &= \sum_{k=2}^{m} \sigma_k(w)\,|z-w|^{2k}.
\end{aligned}$$

With this notation, according to Definition 4.2.1 we have

$$\widetilde{\Delta}\big((z,t),(w,s)\big) = \Phi(z,w) + \Psi(z,w,t,s)^2. \tag{4.3.2}$$

In order to study the effect of differentiating $\widetilde{\Delta}$, we examine separately the effect of differentiating $\Phi(z,w)$ and of differentiating $\Psi^2(z,w,t,s)$. The first will be somewhat easier since the variables t and s do not appear, and so the effect of Z or $\overline{Z}$ is just differentiating with respect to the variables z, $\bar{z}$, w and $\overline{w}$.

Lemma 4.3.2. *If $Q(Z,\overline{Z})$ is a non-commuting polynomial of degree N with each vector field acting either in the (z,t) variables or the (w,s) variables, there is a constant C_Q so that*

$$\big|Q(Z,\overline{Z})\big(\Phi(z,w)\big)\big| \leq C_Q \sum_{k=2}^{m} \sigma_k(w)\,|z-w|^{2k-N}.$$

Proof. Since Φ is independent of t and s, applying Z or $\overline{Z}$ amounts to differentiating with respect to one of z, $\bar{z}$, w or $\overline{w}$. It is easy to see by induction that $Q(Z,\overline{Z})\big(\Phi\big)(z,w)$ is a sum of terms, each of which is a universal constant times an expression of the form

$$\frac{\partial^{r+s}\sigma_k}{\partial w^r\,\partial \overline{w}^s}(w)\,(z-w)^{k-\alpha}\,(\bar{z}-\overline{w})^{k-\beta}$$

where $r + s + \alpha + \beta = N$. The size of this term is dominated by

$$\left|\frac{\partial^{r+s}\sigma_k}{\partial w^r\,\partial\overline{w}^s}(w)\right|\,|z-w|^{2k-\alpha-\beta}.$$

According to Proposition 4.2.3, this is dominated by a sum of terms of the form

$$\sqrt{\sigma_p(w)\,\sigma_q(w)}\,|z-w|^{2k-\alpha-\beta} \tag{4.3.3}$$

where $p + q = 2k + r + s$. Note that $2k - \alpha - \beta = p + q - r - s - \alpha - \beta = p + q - N$. By the Schwarz inequality, the expression in (4.3.3) is dominated by

$$\sigma_p(w)\,|z-w|^{2p-N} + \sigma_q(w)\,|z-w|^{2q-N}.$$

This completes the proof. q.e.d.

We next want to apply the vector fields Z and $\overline{Z}$ to Ψ, but we want to distinguish between the action on the (z,t) variables or the (w,s) variables. Thus write Z_1 and $\overline{Z}_1$ for the operators Z and $\overline{Z}$ acting on (z,t), and Z_2 and $\overline{Z}_2$ for the operators acting on (w,s).

Simple differentiation gives the formulas:

$$Z_2\big(\Psi\big) = +\imath\left[\sum_{j=1}^{m}\frac{1}{j!}\,P_{(1,j)}(w)\,(\bar{z}-\overline{w})^j\right]$$

$$\overline{Z}_2\big(\Psi\big) = -\imath\left[\sum_{j=1}^{m}\frac{1}{j!}\,P_{(j,1)}(w)\,(z-w)^j\right]$$

$$Z_1\big(\Psi\big) = +\imath\left[P_{(1,0)}(z) - \sum_{j=0}^{m-1}\frac{1}{j!}\,P_{(j+1,0)}(w)\,(z-w)^j\right] \tag{4.3.4}$$

$$\overline{Z}_1\big(\Psi\big) = -\imath\left[P_{(0,1)}(z) - \sum_{j=0}^{m-1}\frac{1}{j!}\,P_{(0,j+1)}(w)\,(\bar{z}-\overline{w})^j\right]. \tag{4.3.5}$$

We can also simplify the expressions for $Z_1\big(\Psi\big)$ and $\overline{Z}_1\big(\Psi\big)$. If we expand the polynomial $P_{(1,0)}$ about the point w we obtain

$$
\begin{aligned}
P_{(1,0)}(z) &= \sum_{j=0}^{m-1} \frac{1}{j!} \frac{\partial^j P_{(1,0)}}{\partial w^j}(z-w)^j \\
&\quad + \sum_{k=1}^{m} \sum_{j=0}^{m} \frac{1}{j!\,k!} \frac{\partial^{j+k} P_{(1,0)}}{\partial w^j\, \partial \overline{w}^k}(w)(z-w)^j(\bar{z}-\overline{w})^k \\
&= \sum_{j=0}^{m-1} \frac{1}{j!} P_{(j+1,0)}(z-w)^j \\
&\quad + \sum_{k=1}^{m} \sum_{j=0}^{m} \frac{1}{j!\,k!} P_{(j+1,k)}(w)(z-w)^j(\bar{z}-\overline{w})^k
\end{aligned}
$$

We obtain a similar expression for $P_{(0,1)}$. Inserting these expressions into equations (4.3.4) and (4.3.5) we obtain

$$
\begin{aligned}
Z_1\big(\Psi\big) &= +\imath \left[\sum_{k=1}^{m} \sum_{j=0}^{m} \frac{1}{j!\,k!} P_{(j+1,k)}(w)(z-w)^j(\bar{z}-\overline{w})^k \right]; \\
\overline{Z}_1\big(\Psi\big) &= -\imath \left[\sum_{k=1}^{m} \sum_{j=0}^{m} \frac{1}{j!\,k!} P_{(k,j+1)}(w)(z-w)^k(\bar{z}-\overline{w})^j \right].
\end{aligned}
$$

In summary we have:

Lemma 4.3.3. *The expressions $Z_j\big(\Psi\big)$ and $\overline{Z}_j\big(\Psi\big)$ are independent of t and s, and*

$$
\begin{aligned}
Z_2\big(\Psi\big) &= +\imath \left[\sum_{j=1}^{m} \frac{1}{j!}\, P_{(1,j)}(w)\,(\bar{z}-\overline{w})^j \right]; \\
\overline{Z}_2\big(\Psi\big) &= -\imath \left[\sum_{j=1}^{m} \frac{1}{j!}\, P_{(j,1)}(w)\,(z-w)^j \right]; \\
Z_1\big(\Psi\big) &= +\imath \left[\sum_{k=1}^{m} \sum_{j=0}^{m} \frac{1}{j!\,k!} P_{(j+1,k)}(w)(z-w)^j(\bar{z}-\overline{w})^k \right]; \\
\overline{Z}_1\big(\Psi\big) &= -\imath \left[\sum_{k=1}^{m} \sum_{j=0}^{m} \frac{1}{j!\,k!} P_{(k,j+1)}(w)(z-w)^k(\bar{z}-\overline{w})^j \right].
\end{aligned}
$$

Note that all the derivatives of P appearing in the conclusion of Lemma 4.3.3 are mixed; they involve both w and $\overline{w}$ derivatives. If we apply the vector fields Z or $\overline{Z}$ to these results, only the differentiation with respect to $z, \bar{z}, w$, and $\overline{w}$ have any effect, and we are in the same situation as in the proof of Lemma 4.3.2. We obtain:

Corollary 4.3.4. *If $Q(Z, \overline{Z})$ is a non-commuting polynomial of degree N with each vector field acting either in the (z,t) variables or the (w,s) variables, then $Q(Z,\overline{Z})\big(\Psi\big)$ is a sum of terms, each of which is dominated in absolute value by a constant times $\sqrt{\sigma_p(w)}|z-w|^{p-N}$, with $2 \le p \le m$.*

Finally, we apply $Q(Z,\overline{Z})$ to $|\Psi|^2$. Now either all of the differentiation falls on a single Ψ, giving a term $\Psi\ Q\big(Z,\overline{Z}\big)\big(\Psi\big)$, or we get a sum of products of the form $\big(Q_1(Z,\overline{Z})\big(\Psi\big)\big)\ \big(Q_2(Z,\overline{Z})\big(\Psi\big)\big)$ where the degree of Q_1 plus the degree of Q_2 is N. Using Corollary 4.3.4, this product can be written as a sum of terms, each of which is a constant times $\sqrt{\sigma_p(w)}\,\sqrt{\sigma_q(w)}|z-w|^{p+q-N}$. Thus we have:

Lemma 4.3.5. *If $Q(Z,\overline{Z})$ is a non-commuting polynomial of degree N with each vector field acting either in the (z,t) variables or the (w,s) variables, then there is a constant C_Q so that*

$$\big|Q(Z,\overline{Z})\big(\Psi^2\big)\big| \le\ C_Q\left[\big|\Psi(z,w,t,s)\big|\sum_{k=2}^{m}\sqrt{\sigma_k(w)}|z-w|^{k-N} + \sum_{k=2}^{m}\sigma_k(w)|z-w|^{2k-N}\right].$$

Putting Lemmas 4.3.2 and 4.3.5 together, we obtain:

Lemma 4.3.6 (Main estimate). *If $Q(Z,\overline{Z})$ is a non-commuting polynomial of degree N with each vector field acting either in the (z,t) variables or the (w,s) variables, then there is a constant C_Q so that*

$$\begin{aligned}&\big|Q(Z,\overline{Z})\big(\widetilde{\Delta}\big)\big|\\ &\le C_Q\left[\big|\Psi(z,w,t,s)\big|\sum_{k=2}^{m}\sqrt{\sigma_k(w)}|z-w|^{k-N} + \sum_{k=2}^{m}\sigma_k(w)|z-w|^{2k-N}\right].\end{aligned}$$

Proof of Theorem 4.3.1. We will need estimates on the non-isotropic distance $d\big((z,t),(w,s)\big)$. We have:

Proposition 4.3.7.

$$d\big((z,t),(w,s)\big) \approx \begin{cases} |z-w|, \quad \textit{or} \\ \min\limits_{2\leq k\leq m} \Big\{|\Psi(z,w,t,s)|^{\frac{1}{k}}\, \sigma_k(w)^{-\frac{1}{2k}}\Big\} \end{cases}$$

depending on whether $|\Psi(z,w,t,s)|$ *is smaller or larger than* $\Lambda\big(w,|z-w|\big)$.

Proof. We have

$$d\big((z,t),(w,s)\big) \approx \max\big\{|z-w|,\, \mu\big(w,\big|\Psi(z,w,t,s)\big|\big)\big\}.$$

Thus

$$d\big((z,t),(w,s)\big) \approx \begin{cases} |z-w| & \text{if } \ |\Psi(z,w,t,s)| \\ & \leq \Lambda\big(w,|z-w|\big), \\ \\ \mu\big(w,\big|\Psi(z,w,t,s)\big|\big) & \text{if } \ |\Psi(z,w,t,s)| \\ & \geq \Lambda\big(w,|z-w|\big). \end{cases}$$

Since $\mu\big(w,\big|\Psi(z,w,t,s)\big|\big) \approx \tau\big(w,\Psi(z,w,t,s)^2\big)$, the only issue is showing that for all $w\in\mathbb{C}$ and all $t\geq 0$ we have (uniformly)

$$\tau\big(w,t\big) \approx \min_{2\leq k\leq m} \Big\{t^{\frac{1}{2k}}\, \sigma_k(w)^{-\frac{1}{2k}}\Big\}.$$

Recall that $\sigma\big(w,\tau(w,t)\big)=t$, and so

$$\sum_{k=2}^{m} \sigma_k(w)\,\tau(w,t)^{2k} = t. \tag{4.3.6}$$

It follows that for all $2\leq k\leq m$ we have $\sigma_k(w)\,\tau(w,t)^{2k}\leq t$ and hence

$$\tau(w,t) \leq t^{\frac{1}{2k}}\, \sigma_k(w)^{-\frac{1}{2k}}.$$

Thus

$$\tau(w,t) \leq \min_{2\leq k\leq m} \big\{t^{\frac{1}{2k}}\, \sigma_j(w)^{-\frac{1}{2k}}\big\}.$$

On the other hand, it follows from equation (4.3.6) that there exists an index k_0 with $2\leq k_0\leq m$ so that

$$\sigma_{k_0}(w)\,\tau(w,t)^{2k_0} \geq \frac{1}{m}\, t.$$

Hence

$$\tau(w,t) \geq \left[\frac{1}{m}\right]^{\frac{1}{2k_0}} t^{\frac{1}{2k_0}}\,\sigma_{k_0}(w)^{-\frac{1}{2k_0}} \geq \frac{1}{m} t^{\frac{1}{2k_0}}\,\sigma_{k_0}(w)^{-\frac{1}{2k_0}}.$$

This completes the proof of Proposition 4.3.7. q.e.d.

Proposition 4.3.8.

$$d\big((z,t),(w,s)\big)^N\,\sigma_k(w)\,|z-w|^{2k-N} \lesssim \widetilde{\Delta}\big((z,t),(w,s)\big).$$

Proof. There are two cases to consider. If $\Lambda\big(w,|z-w|\big) \leq |\Psi|$, then according to Proposition 4.3.7, we have for all $2 \leq k \leq m$

$$d\big((z,t),(w,s)\big) \lesssim \big|\Psi(z,w,t,s)\big|^{\frac{1}{k}}\,\sigma_k(w)^{-\frac{1}{2k}}.$$

Hence

$$\begin{aligned}
&d\big((z,t),(w,s)\big)^N\,\sigma_k(w)\,|z-w|^{2k-N}\\
&\qquad\lesssim \big|\Psi(z,w,t,s)\big|^{\frac{N}{k}}\,\sigma_k(w)^{-\frac{N}{2k}}\,\sigma_k(w)^{\frac{N}{2k}}\left[\sigma_k(w)\,|z-w|^{2k}\right]^{1-\frac{N}{2k}}\\
&\qquad\leq \big|\Psi(z,w,t,s)\big|^{\frac{N}{k}}\,\Phi(z,w)^{1-\frac{N}{2k}}\\
&\qquad\leq \big|\Psi(z,w,t,s)\big|^{2} + \Phi(z,w) = \widetilde{\Delta}\big((z,t),(w,s)\big).
\end{aligned}$$

On the other hand if $\Lambda\big(w,|z-w|\big) \geq \big|\Psi\big|$ we have $d\big((z,t),(w,s)\big) \approx |z-w|$, so that

$$d\big((z,t),(w,s)\big)^N\,\sigma_k(w)\,|z-w|^{2k-N} \lesssim \sigma_k(w)\,|z-w|^{2k} \leq \widetilde{\Delta}\big((z,w),(w,s)\big).$$

This completes the proof. q.e.d.

Proposition 4.3.9.

$$d\big((z,t),(w,s)\big)^N\,\big|\Psi(z,w,t,s)\big|\,\sqrt{\sigma_k(w)}|z-w|^{k-N} \leq \widetilde{\Delta}\big((z,t),(w,s)\big).$$

Proof. By the Schwarz inequality, we have

$$\begin{aligned}
&d\big((z,t),(w,s)\big)^N\,\big|\Psi(z,w,t,s)\big|\,\sqrt{\sigma_k(w)}|z-w|^{k-N}\\
&\qquad\leq \big|\Psi(z,w,t,s)\big|^2 + d\big((z,t),(w,s)\big)^{2N}\,\sigma_k(w)\,|z-w|^{2k-2N}\\
&\qquad\leq \widetilde{\Delta}\big((z,t),(w,s)\big)
\end{aligned}$$

by Proposition 4.3.8 and the definition of $\widetilde{\Delta}$. This completes the proof. q.e.d.

With these last two propositions, the proof of Theorem 4.3.1 is complete. q.e.d.

4.4 Construction and properties of the function D

Definition 4.4.1. Set

$$\widetilde{D}\big((z,t),(w,s)\big) = \tau\big(w, \widetilde{\Delta}\big((z,t),(w,s)\big)\big).$$

Remark 4.4.2. It is clear from Proposition 4.2.6 that the function $\widetilde{D}$ is equivalent to the function d. Also since the function $\tau(w,s)$ is smooth on $M \times (0,\infty)$, the function $\widetilde{D}$ is smooth on $(M \times M) -$ diagonal $(M \times M)$.

Proposition 4.4.3. *Regard $\tau = \tau(w,r)$ as a function of w, $\overline{w}$ and $r > 0$. Let $M(Z, \overline{Z})$ be a non-commuting monomial of degree N in the vector fields Z and $\overline{Z}$, each acting either in the (z,t) variables or the (w,s) variables. Then the derivative $M(Z, \overline{Z})\Big(\widetilde{D}\big((z,t),(w,s)\big)\Big)$ is a sum of constants times terms of the form*

$$\partial_w^\alpha \partial_{\overline{w}}^\beta \partial_r^k \tau\big(w, \widetilde{\Delta}\big((z,t),(w,s)\big)\big) \prod_{j=1}^{k} M_j\big(Z, \overline{Z}\big)\widetilde{\Delta}\big((z,t),(w,s)\big)$$

where $N = \alpha + \beta + m_1 + \cdots + m_k$, and each $M_j(Z, \overline{Z})$ is a non-commuting monomial of degree m_j.

Proof. This follows easily by induction on N. q.e.d.

Lemma 4.4.4. *For all integers α, β and j there is a constant $C_{\alpha,\beta,j}$ such that for $w \in \mathbb{C}$ and $s > 0$ we have*

$$\big|\partial_w^\alpha \, \partial_{\overline{w}}^\beta \, \partial_s^j \tau(w,s)\big| \leq C_{\alpha,\beta,j} \, \frac{\tau(w,s)}{s^{j+\alpha+\beta}}.$$

Proof. We first observe that the lemma is true with the function τ replaced by the function σ. This follows easily from the fact that σ is a polynomial with non-negative coefficients, and from Equation (4.2.1). Next, we argue that the same is true for the inverse function for σ by implicit differentiation and an induction argument. q.e.d.

Definition 4.4.5. Set

$$D\big((z,t),(w,s)\big) = \widetilde{D}\big((z,t),(w,s)\big) + \widetilde{D}\big((w,s),(z,t)\big).$$

Theorem 4.4.6. *The function D is smooth on $(M \times M)-$ diagonal $(M \times M)$. It is symmetric:*

$$D\big((z,t),(w,s)\big) = D\big((w,s),(z,t)\big).$$

It is comparable to the non-isotropic distance d. There are positive constants C_1 and C_2 such that for all points $p \neq q \in M$

$$C_1 \leq \frac{D(p,q)}{d(p,q)} \leq C_2.$$

Finally, if $M(Z, \overline{Z})$ be a non-commuting monomial of degree N in the vector fields Z and $\overline{Z}$, each acting either in the (z,t) variables or the (w,s) variables. Then there is a constant C_N so that

$$M(Z, \overline{Z}) D\big((z,t),(w,s)\big) \leq C_N \, D\big((z,t),(w,s)\big)^{1-N}. \tag{4.4.7}$$

Proof. The symmetry is clear from the definition, and we have already established that $\widetilde{D} \approx d$. The differential inequality (4.4.7) follows from Proposition 4.4.3, from Lemma 4.4.4, and from Theorem 4.3.1. Thus the proof is complete. q.e.d.

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